Trace Norm is Less than the Sum of Column (or Row) Lengths

Jason D. M. Rennie
jrennie@gmail.com

March 11, 2006

The trace norm, $\|X\|_\Sigma$, of a matrix is the sum of its singular values. It can also be defined as the dual to the spectral norm [1],

$$\|X\|_\Sigma = \max_{Y:\|Y\|\le 1} \mathrm{tr}(Y^T X), \tag{1}$$

where $\|Y\|$ is the spectral norm (maximum singular value) and $\mathrm{tr}(\cdot)$ denotes the trace. Note that the maximum can always be achieved by a matrix Y with orthonormal rows or columns. Since $\mathrm{tr}(Y^T X) = \mathrm{tr}(X^T Y)$, it must be that the trace of a matrix is less than or equal to its column or row lengths,

$$\|X^T\|_\Sigma = \|X\|_\Sigma \le \sum_i \sqrt{\sum_j X_{ij}^2}. \tag{2}$$

Using this result, we can use the singular value decomposition of a matrix to bound the contribution to the trace norm of a matrix update. Let $X \in \mathbb{R}^{m\times n}$ be a matrix with singular value decomposition $X = U\Sigma V^T$. Let $x \in \mathbb{R}^{1\times n}$ be a row vector. Let $\bar{X} = \begin{bmatrix} X \\ x \end{bmatrix}$ be the updated matrix. Note that we can choose $\bar{U}$ with unit columns and diagonal matrix $\bar{\Sigma}$ so that $\bar{X}V = \bar{U}\bar{\Sigma}$. Since the trace norm is unitarily invariant, $\|\bar{X}\|_\Sigma = \|\bar{U}\bar{\Sigma}\|_\Sigma$. Applying the above result, we have that the trace norm of the updated matrix is less than the trace of $\bar{\Sigma}$,

$$\|\bar{X}\|_\Sigma \le \sum_i \bar{\Sigma}_{ii}. \tag{3}$$

References

[1] J. D. M. Rennie. The relation between the spectral and trace norms. http://people.csail.mit.edu/jrennie/writing, March 2006.