# Solving $Mx = y$

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**Abstract**

Consider the problem of solving for $x$ in the matrix equation $Mx = y$. We consider the case where $M$ is large enough that taking the inverse of $M$ is impractical. $Mx - y$ is nearly the gradient of a quadratic form function. We rewrite the problem as the minimization of a quadratic form function. The Conjugate Gradients algorithm can then be used to efficiently find a solution.

A quadratic form function,

$$f(x) = \frac{1}{2}x^T Ax - b^T x + c, \tag{1}$$

is minimized by the solution to $Ax = b$ if $A$ is symmetric and positive-definite. Shewchuk's Conjugate Gradients tutorial [1] shows how to use this fact to solve for $x$.

Consider solving $Mx = y$ when $M$ is not symmetric and positive definite. There is no corresponding quadratic form function like there is for $Ax = b$. But, $M^T M$ is symmetric and positive definite. And, the solution to $M^T Mx = M^T y$ minimizes the function

$$g(x) = \frac{1}{2}x^T M^T Mx - (M^T y)^T x + \frac{1}{2}y^T y = \frac{1}{2}(Mx - y)^T (Mx - y). \tag{2}$$

Clearly, if $x$ minimizes $g(x)$ and a solution to $Mx = y$ exists, then $x$ is it. If a solution does not exist, then the minimization of $g(x)$ is arguably a reasonable way to select an $x$.

In conclusion, when $M$ is not symmetric and positive definite, we cannot solve $Mx = y$ directly as we can with $Ax = b$. However, we can use the framework for $Ax = b$ to solve $Mx = y$ by choosing $A = M^T M$ and $b = M^T y$.

## References

[1] Jonathan Richard Shewchuk. An introduction to the conjugate gradient method without the agonizing pain. http://www.cs.cmu.edu/~jrs/jrspapers.html, 1994.