The Gaussian-like Normalization Constant

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Abstract

Here we give the procedure for deriving the normalization constant for a Gaussian-like distribution

Consider the function described by

$$\tilde{P}(x) = \exp\left(-x^2\right). \tag{1}$$

Clearly this can be normalized to a distribution since $\tilde{P}(x) < \exp(-x)$ for $|x| \geq 1$ and $\tilde{P}(x) \leq 1$ for $|x| \leq 1$. In fact, $\tilde{P}(x)$ is (nearly) the uni-variate un-normalized standard Normal distribution (Gaussian with zero mean and unit variance).

The standard normal is a very well known distribution, but calculation of the normalization constant isn't completely trivial. The normalization constant is $\tilde{P}$ integrated from $-\infty$ to $+\infty$. But, no antiderivative exists for $\exp(-x^2)$.

The trick is to consider the bi-variate version of $\tilde{P}$ and to integrate using polar coordinates. Define

$$\tilde{Q}(x, y) = \tilde{P}(x)\tilde{P}(y) = \exp(-x^2)\exp(-y^2) = \exp(-(x^2 + y^2)). \tag{2}$$

To normalize $\tilde{Q}$, we must calcualte $Z_Q = \iint \exp(-x^2 - y^2)dxdy$, where both integrals range over $\mathbb{R}$. As is well known, any bi-variate integral over Euclidean coordinates can be rewritten using polar coordinates (e.g. §17.3 of [1]),

$$\int\int f(x, y)dydx = \int\int f(r\cos\theta, r\sin\theta)rdrd\theta. \tag{3}$$

In our case, $f(r\cos\theta, r\sin\theta) = \exp(-r^2)$, so the normalization constant can be written as

$$Z_Q = \int_0^{2\pi}\int_0^{\infty} \exp(-r^2)rdrd\theta. \tag{4}$$

Since the inner integral is constant as a function of θ, we can rewrite it as $Z_Q = 2\pi\int_0^{\infty}\exp(-r^2)rdr$. And, unlike the original integral we posed, this integral is trivial and has an immediately apparante anti-derivative. $d\left(\exp(-r^2)\right) = -2r\exp(-r^2)dr$, so

$$Z_Q = -\pi\left[e^{-\infty} - e^{-0}\right] = \pi. \tag{5}$$

We're almost done. Define $Q(x,y) = \frac{\tilde{Q}(x,y)}{Z_Q} = \frac{1}{\sqrt{\pi}} \exp(-x^2) \frac{1}{\sqrt{\pi}} \exp(-y^2)$, which is a distribution. Due to symmetry, it must be that the normalization constant for $\tilde{P}$ is $\sqrt{\pi}$. Thus, $P(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2)$ is a distribution.

It is straightforward to extend this result to multiple dimensions. Let $\vec{x} \in \mathbb{R}^n$. Then

$$R(\vec{x}) = \pi^{-n/2} \exp(-\|\vec{x}\|_2^2), \tag{6}$$

is a distribution, where $\|\vec{x}\|_2^2 = \sum_i x_i^2$ is the squared L_2-norm.

Thanks to John for reminding me of the polar coordinates trick.

References

[1] E. W. Swokowski. *Calculus*. PWS-Kent Publishing Company, 1983.