Newton's Method

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Let $f(x)$ be a single-variable function. Consider the problem of finding some x such that $f(x) = 0$. If we cannot solve for x analytically, we can still solve for x numerically if f is convex or if we start with a value of x that is sufficiently close to a point where $f(x) = 0$.

Let x_0 be the initial guess. For each iteration, we will use a linear approximation of f to obtain a new estimate. That is, $f(x_{t+1}) = f(x_t) + f'(x_t)(x_{t+1} - x_t)$. Substituting $f(x_{t+1}) = 0$ and solving for x_{t+1} gives the update equation,

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}. \tag{1}$$

Repeating this update usually converges rapidly to a solution.

A useful application of Newton's method is to the problem of finding minima of a function. Let $f(x)$ be a strictly convex, single-variable function with domain being the entire real number line. Then the global minimum must be unique and must occur where $f'(x) = 0$. Using Newton's method, we can iteratively find a numerical solution using the update equation,

$$x_{t+1} = x_t - \frac{f'(x_t)}{f''(x_t)}. \tag{2}$$

We can extend this to the two-variable case. Let $f(x, y)$ be a two-variable function for which we would like to find critical points. In other words, we have a set of two equations that we would like to satisfy, $f_x(x, y) = 0$ and $f_y(x, y) = 0$. Let (x_0, y_0) be an initial guess. For an incremental guess, we approximate f_x and f_y as linear functions. That is, we solve the set of equations,

$$\left[\begin{array}{cc} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{array}\right] \left[\begin{array}{c} \Delta x \\ \Delta y \end{array}\right] = \left[\begin{array}{c} -f_x(x_t, y_t) \\ -f_y(x_t, y_t) \end{array}\right], \tag{3}$$

where $\Delta x = x_{t+1} - x_t$ and $\Delta y = y_{t+1} - y_t$.