Some Linear Algebra Notes

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Definition 1 *A matrix A is positive semi-definite (PSD) iff for all vectors $\vec{x}$, $\vec{x}^T A\vec{x} \geq 0$.*

Definition 2 *The conjugate of a complex number $z = a + bi$ is $\overline{z} = a - bi$.*

Definition 3 *A square matrix A is Hermitian if $A = A^H$, that is $A_{ij} = \overline{A}_{ji}$. For real matrices, Hermitian and symmetric are equivalent (and $A^H \equiv A^T$).*

Theorem 1 *A matrix A is PSD iff $\exists$ B such that $A = B^H B$.*

Theorem 2 *A matrix A is PSD iff $\exists$ a Hermitian matrix C such that $A = C^2$.*

Theorem 3 *If A is PSD then A^{-1} exists and is PSD.*

Theorem 4 *If A is PSD, then $\forall$ integer $k > 0$ $\exists$ a unique PSD matrix B with $A = B^k$. B also satisfies (1) $AB = BA$, (2) $B = p(A)$ for some polynomial p, (3) rank$(B) =$ rank(A), and (4) if A is real then so is B.*

Theorem 5 *The matrix $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ is PSD iff $b^2 < 4ac$.*

Theorem 6 *A real matrix A is PSD iff its symmetric part $B = (A + A^T)/2$ is PSD. Indeed $\vec{x}^T A\vec{x} = \vec{x}^T B\vec{x}$ $\forall \vec{x}$.*

Definition 4 *λ is an eigenvalue of a square matrix A iff for some non-zero $\vec{x}$, $A\vec{x} = \lambda\vec{x}$. $\vec{x}$ is called an eigenvector with corresponding eigenvalue λ. We define $\lambda_i(A)$ to be the i^{th} largest eigenvalue of the matrix A.*

Definition 5 *The trace of a square matrix is the sum of its diagonal elements, tr$(A) = \sum_i A_{ii}$.*

Theorem 7 *Let A be square. Then tr$(A) = \sum_i \lambda_i(A)$.*

Theorem 8 *The eigenvalues of a Hermitian matrix are all real.*

Definition 6 *The spectral norm of a matrix A is the square root of the largest eigenvalue of $A^H A$, $\|A\|_2 = \sqrt{\lambda_1(A^H A)} = \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}$.*

Definition 7 *Let $\|\vec{x}\|$ be a vector norm. The induced matrix norm is $\|A\| = \max_{\|\vec{x}\|=1} \|A\vec{x}\|$. The spectral norm is the L_2 induced matrix norm.*