Max-Antecedent is Convex Plus Concave

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Abstract

We show that the Max-Antecedent objective is a convex function plus a concave function.

- Let $W \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^d$ be indexed by examples, then classes, then components. W is assumed fixed and known.
- Let $\vec{y} = \{y_1, \ldots, y_n\}$, $y_i \in \{1, \ldots, m\}$, be the class labels of the examples.
- Let $\vec{x} \in \mathbb{R}^d$ be the parameter vector.
- Let $Z_i(\vec{x}) = \sum_{j=1}^m \max_{k \in \{1,\ldots,l\}} \exp\left(W_{ijk}^T \vec{x}\right)$. Our objective is

$$J(\vec{x}) = \prod_{i=1}^n \frac{1}{Z_i(\vec{x})} \exp\left(\max_{k \in \{1,\ldots,l\}} W_{iy_ik}^T \vec{x}\right) \tag{1}$$

$$\log J(\vec{x}) = \sum_{i=1}^n \max_{k \in \{1,\ldots,l\}} W_{iy_ik}^T \vec{x} - \log Z_i(\vec{x}) \tag{2}$$

Clearly, the first term is convex (the maximum of a set of linear functions). $Z_i(\vec{x})$ is convex—the maximum of a set of convex functions. The negation of a convex function is concave. Sums of convex functions are convex; sums of concave functions are concave. So, $\log J(\vec{x})$ is a convex function plus a concave function.